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Asymptotically Schwarzschild Spacetimes

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It is shown that if an asymptotically flat spacetime is asymptotically stationary, in the sense that $\pounds_\xi g_{ab}$ vanishes at the rate $\sim t^{-3}$ for asymptotically timelike vector field ξ^a, and the energy-momentum tensor vanishes at the rate $\sim t^{-4}$, then the spacetime is an asymptotically Schwarzschild spacetime. This gives a new aspect of the uniqueness theorem of a black hole.

I. INTRODUCTION

There are many astrophysical phenomena that are best explained by black holes, for example, active galactic nuclei and X-ray binaries. The analysis on the phenomena are done assuming that those black holes are described by the Kerr spacetimes. This is because the uniqueness theorem of a black hole guarantees that a spacetime which is stationary, vacuum and asymptotically flat is uniquely the Kerr spacetime, and we believe that when gravitational collapse takes place and a black hole is formed, the spacetime around it becomes vacuum and accordingly stationary. (For the details on the uniqueness theorem of a black hole, see [1].) However, one may argue that such a spacetime does not become *exactly* vacuum nor *exactly* stationary: the spacetime becomes *asymptotically* vacuum, and accordingly *asymptotically* stationary at a certain rate of the time. In this context, a more adequate "uniqueness theorem" is the one that states an asymptotically stationary, vacuum and flat spacetime is uniquely an asymptotically Schwarzschild spacetime. This is what we show in this paper. (One may find it peculiar that an asymptotically *stationary* spacetime is an asymptotically Schwarzschild spacetime. However, if we define *asymptotically* Schwarzschild spacetimes as a class of spacetimes that asymptotically approach the exact Schwarzschild spacetime, the spacetimes comprise a wide class of spacetimes, including the Kerr spacetime, which is stationary. See the end of sec.IV for the details.)

To show the theorem, we use the notion of the asymptotic flat spacetime, first introduced by Ashtekar and Romano [2] at spacelike infinity and succeedingly developed in our previous study [3] at timelike infinity. We investigate the asymptotic behavior of the gravitational field at the future timelike infinity, because we would like to know whether the gravitational field approaches asymptotically that of the Schwarzschild spacetime at the *late time* when the spacetime becomes asymptotically stationary. The standard definition of the asymptotic flatness [4] is based on the conformal completion method, which is used to obtain the Penrose diagram. This method is useful in that spacelike and timelike infinities can be simultaneously treated with null infinity, and thus that investigation concerning the global causal structure can be done. However, the method compresses the spacelike and timelike infinities, which possess rich 3-manifold information, down to points, and this compression results in a complicated differential structure at these

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points and makes it difficult to obtain the comprehensive picture of the behavior of gravitational fields in general asymptotically flat spacetimes. In contrast, the completion method introduced by [2] for defininining the asymptotic flatness at spacelike infinity leaves the infinity as a 3-manifold. As a result, the complicated differential structure in the former treatment can be avoided. Subsequently, in our previous study [3], we applied the method to timelike infinity and clarified that the method leads to a definite picture of hierarchy in the asymptotic behavior of the gravitational field and the symmetry, and that it is suitable to discuss such a notion as an “asymptotically Schwarzschild spacetime”. (Perng also investigates hierarchy in the asymptotic gravitational field at spacelike infinity [5], although it does not directly correspond to the hierarchy discussed in [3].) In this paper, we further investigate the hierarchy and prove the theorem stating that an asymptotically stationary, vacuum and flat spacetime is uniquely an asymptotically Schwarzschild spacetime.

The plan of this paper is the following. Sec.II is devoted to preliminaries. The definition of the asymptotic flatness at timelike infinity is recalled, and some of its useful consequences are summarized for the subsequent discussion. In sec.III, the first order asymptotic structure is explored, that is, the one order higher structure than the basic asymptotic structure that all the asymptotic flat spacetimes possess. Then, in sec.IV, we introduce the notion of asymptotic stationarity and investigate how the condition that an asymptotic flat spacetime be asymptotically stationary, constrain the asymptotic structure. The investigation leads to the proof of the main theorem.

Throughout the paper, we follow the notation of Wald [6].

II. PRELIMINARIES

In this section, we recall the definition of asymptotic flatness at timelike infinity of [3] that will be used in the main proof and fix the notation.

DEFINITION: A physical spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is said to possess an *asymptote at future timelike infinity* $\breve{\imath}^+$ *to order* n (ATI-n) for a non-negative integer n, if there exists a manifold $\mathcal{M}$ with boundary $\mathcal{H}$, a smooth function Ω defined on $\mathcal{M}$, and an imbedding Ψ of an open subset $\hat{\mathcal{F}}$ in $\hat{\mathcal{M}}$ to $\mathcal{M}-\mathcal{H}$ satisfying the following conditions:

(1) $\breve{\imath}^+ := \partial\mathcal{F} \cap (\mathcal{M} - \Psi(\hat{\mathcal{M}}))$ is not empty and $\breve{\imath}^+ \subset I^+(\mathcal{F})$ where $\mathcal{F} := \Psi(\hat{\mathcal{F}})$;

(2) $\Omega \breve{=} 0$ and $\nabla_a\Omega \breve{\neq} 0$, where $\breve{=}$ denotes the equality evaluated on $\breve{\imath}^+$;

(3) $n^a := \Omega^{-4}\Psi^*\hat{g}^{ab}\nabla_b\Omega$ and $q_{ab} := \Omega^2(\Psi^*\hat{g}_{ab} + \Omega^{-4}F^{-1}\nabla_a\Omega\nabla_b\Omega)$ admit smooth limits to $\breve{\imath}^+$ with q_{ab} having signature(+++) on $\breve{\imath}^+$, where $F := -\pounds_n\Omega$; and

(4) $\lim_{\to\breve{\imath}^+}\Omega^{-(2+n)}T_{\hat{\mu}\hat{\nu}} \breve{=} 0$ where $\hat{T}_{\hat{\mu}\hat{\nu}} := \Psi^*[(\hat{e}_\mu)^a(\hat{e}_\nu)^b\hat{T}_{ab}]$ in which $\{(\hat{e}_\mu)^a\}$ and $\hat{T}_{ab}$ are a tetrad and the physical energy-momentum tensor of $(\hat{\mathcal{M}}, \hat{g}_{ab})$, respectively.

Henceforth, we use a tetrad consisting of a unit vector field $\hat{n}^a$ that is normal to the Ω-const. surfaces and a triad $\{(\hat{e}_{\rm I})^a\}_{\rm I=1,2,3}$ of the metric $\hat{q}_{ab} := \hat{g}_{ab} + \hat{n}_a\hat{n}_b$ on the Ω-const. surface. We denote the timelike components with the subscript 0 and the spacelike component with capital-Roman-letter subscript, e.g., $\hat{A}^{\hat{0}} := \hat{n}_a\hat{A}^a$ and $\hat{A}^{\hat{\rm I}} := (\hat{e}_{\rm I})_a\hat{A}^a$. If a tensor $A^{a\cdots b}_{\ \ c\cdots d}$ admits a smooth limit to $\breve{\imath}^+$, it is useful to define the n-th order term of $A^{a\cdots b}_{\ \ c\cdots d}$ as

$$ {}^{(0)}A^{a\cdots b}_{\ \ c\cdots d} := \lim_{\to\breve{\imath}^+} A^{a\cdots b}_{\ \ c\cdots d} \qquad \text{and} \qquad {}^{(n)}A^{a\cdots b}_{\ \ c\cdots d} := \lim_{\to\breve{\imath}^+} \Omega^{-n}(A^{a\cdots b}_{\ \ c\cdots d} - \sum_{\ell=0}^{n-1} {}^{(\ell)}A^{a\cdots b}_{\ \ c\cdots d}\,\Omega^\ell) \qquad \text{for } n \geq 1. \tag{1} $$

This definition of the n-th order terms of a tensor implies that in the vicinity of $\breve{\imath}^+$, $A^{a\cdots b}_{\ \ c\cdots d}$ can be expanded as

$$ A^{a\cdots b}_{\ \ c\cdots d} = \sum_{n=0}^{\infty} {}^{(n)}A^{a\cdots b}_{\ \ c\cdots d}\,\Omega^n. \tag{2} $$

Since all the equations appearing in the following discussion are those on $\mathcal{M}$, unless it may cause ambiguity, we omit hereafter Ψ^* in front of the tensors defined on $\hat{\mathcal{M}}$ for brevity.

Before we examine the properties of ATI-n spacetimes, we introduce some valuable tensors. First, the projection operator with respect to the Ω-const. surface can be introduced as

$$ q^a{}_b := \sum_{I=1,2,3} (\hat{e}_{\rm I})^a(\hat{e}_{\rm I})_b = \delta^a{}_b + F^{-1}n^a\nabla_b\Omega. $$

This operator admits a smooth limit to $\breve{\imath}^+$ by virtue of the definition of an ATI-n spacetime. Second, note that the above definition of an ATI-n spacetime implies that $\breve{\imath}^+$ is a 3-submanifold of $\mathcal{M}$ with an imbedding, say Π. Hence,

if a tensor field $A^{a\cdots b}{}_{c\cdots d}$ is tangential to $\breve{\imath}^{+}$, or $A^{a\cdots b}{}_{c\cdots d} := q^a{}_e \cdots q^b{}_f\, q^g{}_c \cdots q^h{}_d A^{e\cdots f}{}_{g\cdots h}$, it is useful to consider the tensor field $\Pi^* A^{a\cdots b}{}_{c\cdots d}$ defined on $\breve{\imath}^{+}$. Hereafter, we denote such a tensor field in boldface, i.e., $\boldsymbol{A}^{a\cdots b}{}_{c\cdots d} := \Pi^* A^{a\cdots b}{}_{c\cdots d}$, and say that $A^{a\cdots b}{}_{c\cdots d}$ induces $\boldsymbol{A}^{a\cdots b}{}_{c\cdots d}$ on $\breve{\imath}^{+}$.

Now we explore the consequence of the definition of an ATI-n spacetime for $n = 0$. Solving the Einstein equation under the fall-off condition on the energy-momentum tensor, $\lim_{\to\breve{\imath}^{+}} \Omega^{-2}\hat{T}_{\hat{\mu}\hat{\nu}} = 0$, it is found that

$$\boldsymbol{F} \breve{=} 1 \qquad \text{and} \qquad \boldsymbol{q}_{ab} \breve{=} \boldsymbol{h}_{ab} \tag{3}$$

in an ATI-0 spacetime, where $h_{ab} = (d\chi)_a(d\chi)_b + \sinh^2\chi[(d\theta)_a(d\theta)_b + \sin^2\theta(d\phi)_a(d\phi)_b]$ is the 3-metric of the unit timelike 3-hyperboloid. (For the details of the derivation, see [3].) Because eq.(3) is a gravitational structure common to all the ATI-0 spacetimes, we call it the *zero-th order asymptotic structure*. Using conformal time $\eta := \ln\Omega$, these results imply

$$\hat{g}_{ab} = {}^{(0)}\hat{g}_{ab} + \Omega\,{}^{(1)}\hat{g}_{ab} + O(\Omega^2) \qquad \text{where} \qquad {}^{(0)}\hat{g}_{ab} = (e^{-\eta})^2\left[-(d\eta)_a(d\eta)_b + h_{ab}\right] \tag{4}$$

in which ${}^{(n)}\hat{g}_{ab}$ is defined by

$${}^{(n)}\hat{g}_{ab} := \sum_{\mu,\nu} (\hat{e}_\mu)_a(\hat{e}_\nu)_b\, {}^{(n)}g_{\hat{\mu}\hat{\nu}} \tag{5}$$

with a function $g_{\hat{\mu}\hat{\nu}} := (\hat{e}_\mu)^a(\hat{e}_\nu)^b\hat{g}_{ab}$ that admits a smooth limit to $\breve{\imath}^{+}$ and thus is expanded in the manner described in eq.(1). ${}^{(0)}\hat{g}_{ab}$ is a metric of the Milne universe and is equivalent to the metric of a Minkowski spacetime, $\hat{g}^{\mathsf{MIN}}_{ab}$. In other words, eq.(4) tell us that an ATI-0 spacetime is an asymptotically Minkowski spacetime:

$$\hat{g}_{ab} = \hat{g}^{\mathsf{MIN}}_{ab} + O(\Omega) \tag{6}$$

Hence, it is no surprise that the Riemann tensor asymptotically vanishes in such a spacetime. The trace part of the Riemann tensor asymptotically vanishes by virtue of the fall-off condition on the energy-momentum tensor. The traceless part, or the Weyl tensor $\hat{C}_{ambn}$, can be best investigated by decomposing the tensor into the electric part $\hat{E}_{ab} := \hat{C}_{ambn}\hat{n}^m\hat{n}^n$ and the magnetic part $\hat{B}_{ab} := {}^*\hat{C}_{ambn}\hat{n}^m\hat{n}^n$ where ${}^*\hat{C}_{ambn}$ denotes the dual of the 2-form $\hat{C}_{[am]bn}$. In terms of $\hat{q}_{ab}$ and $\hat{n}^a$, $\hat{E}_{ab}$ and $\hat{B}_{ab}$ are given by

$$\begin{aligned}
\hat{E}_{ab} &= \hat{K}_{ar}\hat{K}^r{}_b - \pounds_{\hat{n}}\hat{K}_{ab} + \hat{D}_{(a}\hat{a}_{b)} + \hat{a}_a\hat{a}_b + \frac{1}{2}(\hat{q}^r_a\hat{q}^s_b - \hat{q}_{ab}\hat{n}^r\hat{n}^s)\hat{L}_{rs} \\
\hat{B}_{ab} &= \hat{\epsilon}_{ra}{}^s\hat{D}^r\hat{K}_{bs} + \frac{1}{2}\hat{\epsilon}_{ab}{}^r\hat{n}^s\hat{L}_{rs}
\end{aligned} \tag{7}$$

where $\hat{K}_{ab} := \frac{1}{2}\pounds_{\hat{n}}\hat{q}_{ab}$, $\hat{L}_{ab} := \hat{R}_{ab} - \frac{1}{6}\hat{R}\hat{g}_{ab}$, $\hat{a}_a := \hat{q}_{ar}\hat{n}^s\nabla_s\hat{n}^r$, and $\hat{D}_a$ and $\hat{\epsilon}_{abc}$ are the derivative operator and the volume element associated with $\hat{q}_{ab}$, respectively. Using the definitions of n^a and q_{ab}, and imposing the fall-off condition on the energy-momentum tensor, it can be found that both admits a smooth limit to $\breve{\imath}^{+}$ and thus can be expanded by the manner described in eq.(1). Their leading terms are

$${}^{(0)}\boldsymbol{E}_{ab} \breve{=} 0 \qquad \text{and} \qquad {}^{(0)}\boldsymbol{B}_{ab} \breve{=} 0. \tag{8}$$

Simple calculation shows that the behaviors of ${}^{(n)}\boldsymbol{E}_{ab}$ and ${}^{(n)}\boldsymbol{B}_{ab}$ for $n \geq 1$ depends on that of the higher order energy-momentum tensor, which is arbitrary in an ATI-0 spacetime. (See [3] for the derivations.)

III. THE FIRST-ORDER ASYMPTOTIC STRUCTURE

In this section, we derive the first order asymptotic structure, that is, a structure which is possessed by all the ATI-1 spacetimes but not by ATI-0 spacetimes. The part of the structure, i.e., the $O(\Omega)$ term of F, has been already derived in [3]. Here, we treat the whole structure in a systematic way. The point is that the first order asymptotic structure may be considered as the perturbation to the Milne universe, eq.(4), where Ω plays the role of a small parameter of the perturbation. Hence, we can apply the technique of the cosmological perturbation [7,8] to derive the first order asymptotic structure.

The energy-momentum tensor of an ATI-1 spacetime satisfies a stronger fall-off condition, $\lim_{\to\breve{\imath}^{+}} \Omega^{-3}\hat{T}_{\hat{\mu}\hat{\nu}} = 0$, than that of an ATI-0 and thus the behavior of asymptotic gravitational fields is constrained stronger. In other words, ATI-1 spacetimes possess more asymptotic gravitational structure in common. The structure can be derived by solving the

Einstein equation under the condition $\lim_{\to \breve{\imath}^+} \Omega^{-3}\hat{T}_{\hat{\mu}\hat{\nu}} = 0$. To obtain the equation, we first decompose the first-order metric ${}^{(1)}\hat{g}_{ab}$ as

$$ {}^{(1)}\hat{g}_{ab} = (e^{-\eta})^2 \left[{}^{(1)}F(d\eta)_a(d\eta)_b - 2\,{}^{(1)}\beta_{(a}(d\eta)_{b)} - 2\,{}^{(1)}\psi h_{ab} + 2\,{}^{(1)}\chi_{ab} \right] \tag{9} $$

where ${}^{(1)}\beta_a$ and ${}^{(1)}\chi_{ab}$ are tangential to the Ω-const. surfaces, i.e., $q_a{}^b\,{}^{(1)}\beta_b = {}^{(1)}\beta_a$ and $q_a{}^c q_b{}^d\,{}^{(1)}\chi_{cd} = {}^{(1)}\chi_{ab}$; and ${}^{(1)}\chi_{ab}$ is traceless, i.e., $q^{ab}\,{}^{(1)}\chi_{ab} = 0$. With these quantities, the Einstein equation induces on $\breve{\imath}^+$ the following set of differential equations in an ATI-1 spacetime satisfying $\lim_{\to \breve{\imath}^+} \Omega^{-3}\hat{T}_{\hat{\mu}\hat{\nu}} = 0$:

$$ \begin{aligned} 3\,{}^{(1)}\boldsymbol{F} + 2\triangle\,{}^{(1)}\boldsymbol{\psi} - 2\boldsymbol{D}_m\,{}^{(1)}\boldsymbol{\beta}^m + \boldsymbol{D}^m\boldsymbol{D}_n\,{}^{(1)}\boldsymbol{\chi}_m{}^n &\,\breve{=}\, 0 \\ \boldsymbol{D}_a({}^{(1)}\boldsymbol{F} + 2\,{}^{(1)}\boldsymbol{\psi}) + \frac{1}{2}(\triangle - 2)\,{}^{(1)}\boldsymbol{\beta}_a - \frac{1}{2}\boldsymbol{D}_a(\boldsymbol{D}_m\,{}^{(1)}\boldsymbol{\beta}^m) + \boldsymbol{D}_m\,{}^{(1)}\boldsymbol{\chi}_a{}^m &\,\breve{=}\, 0 \\ (\boldsymbol{h}_{ab}\triangle - \boldsymbol{D}_a\boldsymbol{D}_b)({}^{(1)}\boldsymbol{F} + 2\,{}^{(1)}\boldsymbol{\psi}) + 2\boldsymbol{D}_{(a}\,{}^{(1)}\boldsymbol{\beta}_{b)} + 2(\triangle + 3)\,{}^{(1)}\boldsymbol{\chi}_{ab}& \\ -2\boldsymbol{h}_{ab}\boldsymbol{D}_m\,{}^{(1)}\boldsymbol{\beta}^m - 4\boldsymbol{D}_{(a}\boldsymbol{D}^m\,{}^{(1)}\boldsymbol{\chi}_{b)m} + 2\boldsymbol{h}_{ab}\boldsymbol{D}^m\boldsymbol{D}_n\,{}^{(1)}\boldsymbol{\chi}_m{}^n &\,\breve{=}\, 0 \end{aligned} \tag{10} $$

where $\boldsymbol{D}_a$ is the derivative operator associated with the metric $\boldsymbol{h}_{ab}$ of $\breve{\imath}^+$, and $\triangle := \boldsymbol{D}^a\boldsymbol{D}_a$. To simplify the equation above, we consider the Poisson gauge in which ${}^{(1)}\boldsymbol{\beta}_a$ is transverse and ${}^{(1)}\boldsymbol{\chi}_{ab}$ is transverse-traceless. Noting that the quantities are transformed as

$$ \begin{aligned} {}^{(1)}\overline{\boldsymbol{F}} &\,\breve{=}\, {}^{(1)}\boldsymbol{F} & \qquad {}^{(1)}\overline{\boldsymbol{\beta}}_a &\,\breve{=}\, {}^{(1)}\boldsymbol{\beta}_a + \boldsymbol{D}_a\boldsymbol{T} - \boldsymbol{L}_a \\ {}^{(1)}\overline{\boldsymbol{\psi}} &\,\breve{=}\, {}^{(1)}\boldsymbol{\psi} + \boldsymbol{T} - \frac{1}{3}\boldsymbol{D}^a\boldsymbol{L}_a & \qquad {}^{(1)}\overline{\boldsymbol{\chi}}_{ab} &\,\breve{=}\, {}^{(1)}\boldsymbol{\chi}_{ab} + \boldsymbol{D}_{(a}\boldsymbol{L}_{b)} - \frac{1}{3}\boldsymbol{h}_{ab}\boldsymbol{D}_m\boldsymbol{L}^m \end{aligned} \tag{11} $$

under a gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega L^a$, we find that the Poisson gauge can be always chosen if we set

$$ \begin{aligned} \boldsymbol{T} &:\,\breve{=}\, -\tfrac{1}{2}\triangle^{-1}[3(\triangle - 3)^{-1}\boldsymbol{D}^m\boldsymbol{D}^n\,{}^{(1)}\boldsymbol{\chi}_{mn} + 2\boldsymbol{D}^a\,{}^{(1)}\boldsymbol{\beta}_a] \\ \boldsymbol{L}_a &:\,\breve{=}\, \tfrac{1}{2}(\triangle - 2)^{-1}\{\boldsymbol{D}_a[(\triangle - 3)^{-1}\boldsymbol{D}^m\boldsymbol{D}^n\,{}^{(1)}\boldsymbol{\chi}_{mn}] - 4\boldsymbol{D}^m\,{}^{(1)}\boldsymbol{\chi}_{ma}\}, \end{aligned} \tag{12} $$

for the generator $\xi^a = \Omega T(\partial_\eta)^a + \Omega L^a$, which satisfy $\boldsymbol{D}^a\,{}^{(1)}\boldsymbol{\chi}_{ab} + \boldsymbol{D}^a\boldsymbol{D}_{(a}\boldsymbol{L}_{b)} - \frac{1}{3}\boldsymbol{D}_b(\boldsymbol{D}_m\boldsymbol{L}^m) \,\breve{=}\, 0$ and $\boldsymbol{D}^a\,{}^{(1)}\boldsymbol{\beta}_a + \triangle\boldsymbol{T} - \boldsymbol{D}^a\boldsymbol{L}_a \,\breve{=}\, 0$. Note that this Poisson gauge is preserved under the trasformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega L^a$ where T and L_a satisfies

$$ \triangle\boldsymbol{T} - \boldsymbol{D}^m\boldsymbol{L}_m \,\breve{=}\, 0 \quad \text{and} \quad (\triangle - 2)\boldsymbol{L}_b + \frac{1}{3}\boldsymbol{D}_b\boldsymbol{D}_m\boldsymbol{L}^m \,\breve{=}\, 0. \tag{13} $$

In this gauge, the Einstein equation (10) simplifies to

$$ 3\,{}^{(1)}\boldsymbol{F} + 2\triangle\,{}^{(1)}\boldsymbol{\psi} \,\breve{=}\, 0 \tag{14a} $$

$$ \boldsymbol{D}_a({}^{(1)}\boldsymbol{F} + 2\,{}^{(1)}\boldsymbol{\psi}) + \tfrac{1}{2}(\triangle - 2)\,{}^{(1)}\boldsymbol{\beta}^T_a \,\breve{=}\, 0 \tag{14b} $$

$$ (\boldsymbol{h}_{ab}\triangle - \boldsymbol{D}_a\boldsymbol{D}_b)({}^{(1)}\boldsymbol{F} + 2\,{}^{(1)}\boldsymbol{\psi}) + 2\boldsymbol{D}_{(a}\,{}^{(1)}\boldsymbol{\beta}^T_{b)} + 2(\triangle + 3)\,{}^{(1)}\boldsymbol{\chi}^{TT}_{ab} \,\breve{=}\, 0 \tag{14c} $$

where the over-bar is omitted which shows that the quantity is transformed and the subscripts T and TT denote that the quantities are transverse and transverse-traceless, respectively.

First, we derive the scalar ${}^{(1)}\boldsymbol{F}$. Subtracting eq.(14a) from the divergence of eq.(14b), we obtain

$$ (\triangle - 3)\,{}^{(1)}\boldsymbol{F} \,\breve{=}\, 0. \tag{15} $$

The general solution of the above equation can be derived by first expanding the function ${}^{(1)}\boldsymbol{F}$ as

$$ {}^{(1)}\boldsymbol{F}(\boldsymbol{\chi}, \boldsymbol{\theta}, \boldsymbol{\phi}) \,\breve{=}\, \sum \boldsymbol{a}_{\ell m}\,{}^{(1)}\boldsymbol{F}^{\ell m}(\boldsymbol{\chi}, \boldsymbol{\theta}, \boldsymbol{\phi}) \quad \text{where} \quad {}^{(1)}\boldsymbol{F}^{\ell m}(\boldsymbol{\chi}, \boldsymbol{\theta}, \boldsymbol{\phi}) := \boldsymbol{T}^\ell_0(\boldsymbol{\chi})\boldsymbol{Y}^{\ell m}(\boldsymbol{\theta}, \boldsymbol{\phi}) \tag{16} $$

in which the summation is taken over $\ell \in \mathbb{Z}$ and $|m| \leq |\ell|$, and $\boldsymbol{Y}^{\ell m}(\boldsymbol{\theta}, \boldsymbol{\phi})$ are the 2-dimensional spherical harmonics. Substituting eq.(16) into eq.(15), it is found that $\boldsymbol{T}^\ell_0$ is given by

$$\boldsymbol{T}_0^\ell(\chi) = \boldsymbol{\mathcal{P}}_2^\ell(\chi) \tag{17}$$

where the functions $\boldsymbol{\mathcal{P}}_n^\ell(\chi)$ is defined by

$$\boldsymbol{\mathcal{P}}_n^\ell(\chi) := \frac{1}{\sqrt{\sinh \boldsymbol{\chi}}} \boldsymbol{P}_{n-\frac{1}{2}}^{\ell+\frac{1}{2}}(\cosh \boldsymbol{\chi}), \tag{18}$$

and satisfies

$$\boldsymbol{\mathcal{P}}_n^{\ell\,\prime\prime}(\boldsymbol{\chi}) + \frac{2}{\tanh \boldsymbol{\chi}} \boldsymbol{\mathcal{P}}_n^{\ell\,\prime}(\boldsymbol{\chi}) - \left(n^2 - 1 + \frac{\ell(\ell+1)}{\sinh^2 \boldsymbol{\chi}}\right) \boldsymbol{\mathcal{P}}_n^\ell(\boldsymbol{\chi}) \doteqdot 0 \tag{19}$$

where the prime $'$ denotes the derivative with respect to $\boldsymbol{\chi}$; and $P_\nu^\mu(z)$ is an associated Legendre function normalized as

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\nu, \nu+1, 1-\mu; \frac{1-z}{2}\right) \quad \text{for} \quad z > 1. \tag{20}$$

(Note here that unconventional choice of ℓ is taken. It is related to the conventional choice ℓ_{c} by $\ell = -\ell_{\mathsf{c}} - 1$, and thus it is $\ell \le -1$ terms that are regular. The reason for the chioce is to have the coefficinent a_{00} in eq.(16) correspond to the mass of a spacetime. See eq.(51).)

Second, we derive the scalar ${}^{(1)}\boldsymbol{\psi}$. Adding eq.(15) to eq.(14a), we obtain

$$\triangle({}^{(1)}\boldsymbol{F} + 2\,{}^{(1)}\boldsymbol{\psi}) \doteqdot 0. \tag{21}$$

Let $\boldsymbol{A}$ be the solution of $\triangle \boldsymbol{A} \doteqdot 0$. Then, ${}^{(1)}\boldsymbol{\psi}$ may be given by

$${}^{(1)}\boldsymbol{\psi} \doteqdot -\frac{1}{2}\,{}^{(1)}\boldsymbol{F} + \frac{\boldsymbol{A}}{2}. \tag{22}$$

Now, perform the gauge trasfomation generated by $\xi^a = \Omega T (\partial_\eta)^a$ where $\boldsymbol{T} \doteqdot -\boldsymbol{A}/2$. This transformation preserves the Poisson gauge, which can be seen from eq.(13), and simplifies ${}^{(1)}\boldsymbol{\psi}$:

$${}^{(1)}\boldsymbol{\psi} \doteqdot -\frac{1}{2}\,{}^{(1)}\boldsymbol{F}. \tag{23}$$

Next, we consider the vector field ${}^{(1)}\boldsymbol{\beta}_a^T$. Substitution of eq.(23) into eq.(14b) yields

$$(\triangle - 2)\,{}^{(1)}\boldsymbol{\beta}_a^T \doteqdot 0. \tag{24}$$

Hence, together with the fact ${}^{(1)}\boldsymbol{\beta}_a^T$ is transverse, we see from eq.(13) that the gauge trasformation generated by $\xi^a = \Omega L^a$ where $\boldsymbol{L}_a \doteqdot {}^{(1)}\boldsymbol{\beta}_a^T$ preserves the Poisson gauge, and results in

$${}^{(1)}\boldsymbol{\beta}_a^T \doteqdot 0. \tag{25}$$

Finally, we derive the tensor field ${}^{(1)}\boldsymbol{\chi}_{ab}^{TT}$. Substituting eqs.(23)(25) into eq.(14c), we obtain

$$(\triangle + 3)\,{}^{(1)}\boldsymbol{\chi}_{ab}^{TT} \doteqdot 0. \tag{26}$$

To solve this equation, we consider the spherical harmonic expansion again. As generally done, we decompose ${}^{(1)}\boldsymbol{\chi}_{ab}^{TT}$ into the even (or electric-type) parity part ${}^{(1)}\boldsymbol{\chi}_{ab}^{TT(+)}$ and the odd (or magnetic-type) parity part ${}^{(1)}\boldsymbol{\chi}_{ab}^{TT(-)}$: ${}^{(1)}\boldsymbol{\chi}_{ab}^{TT} \doteqdot {}^{(1)}\boldsymbol{\chi}_{ab}^{TT(+)} + {}^{(1)}\boldsymbol{\chi}_{ab}^{TT(-)}$. In the present case, the even parity part that satisfies eq. (26) is found to be [9]

$${}^{(1)}\boldsymbol{\chi}_{ab}^{TT(+)} \doteqdot (\boldsymbol{D}_a \boldsymbol{D}_b - \boldsymbol{h}_{ab})\boldsymbol{X} \tag{27}$$

where $\boldsymbol{X}$ is a function that satisfies $(\triangle - 3)\boldsymbol{X} \doteqdot 0$. On the other hand, the odd parity part is found to be [10]

$${}^{(1)}\boldsymbol{\chi}_{ab}^{TT(-)} \doteqdot \sum_{\ell \neq 0} \boldsymbol{b}_{\ell m}^{(-)}\,{}^{(1)}\boldsymbol{\chi}_{ab}^{TT(-)\ell m} \tag{28}$$

where the summation is taken over $\ell \in \mathbb{Z}$ for $\ell \neq 0$, $|m| \leq |\ell|$; and

$$ {}^{(1)}\!\boldsymbol{\chi}^{TT(-)\ell m}_{\chi\chi} :\breve{=} 0, \quad {}^{(1)}\!\boldsymbol{\chi}^{TT(-)\ell m}_{\chi A} :\breve{=} \boldsymbol{T}^{\ell}_{1}(\boldsymbol{\chi})\boldsymbol{\epsilon}_{A}{}^{B}\boldsymbol{\mathcal{D}}_{B}\boldsymbol{Y}^{\ell m} \quad \text{and} \quad {}^{(1)}\!\boldsymbol{\chi}^{TT(-)\ell m}_{AB} :\breve{=} \boldsymbol{T}^{\ell}_{2}(\boldsymbol{\chi})\boldsymbol{\epsilon}^{C}{}_{(A}\boldsymbol{\mathcal{D}}_{B)}\boldsymbol{\mathcal{D}}_{C}\boldsymbol{Y}^{\ell m} \tag{29} $$

in which functions $\boldsymbol{T}^{\ell}_{1}(\boldsymbol{\chi})$ and $\boldsymbol{T}^{\ell}_{2}(\boldsymbol{\chi})$ are given by

$$ \boldsymbol{T}^{\ell}_{1}(\boldsymbol{\chi}) \breve{=} \boldsymbol{\mathcal{P}}^{\ell}_{0}(\boldsymbol{\chi}) \qquad \text{and} \qquad \boldsymbol{T}^{\ell}_{2}(\boldsymbol{\chi}) \breve{=} \frac{\sinh^2 \boldsymbol{\chi}}{(\ell-1)(\ell+2)}\left[\partial_{\chi} + 2\frac{\cosh \boldsymbol{\chi}}{\sinh \boldsymbol{\chi}}\right]\boldsymbol{\mathcal{P}}^{\ell}_{0}(\boldsymbol{\chi}) \quad \text{for } \ell \neq 1, -2. \tag{30} $$

(Here, there is no need to derive $\boldsymbol{T}^{\ell}_{2}(\boldsymbol{\chi})$ for $\ell = 1$ or $\ell = -2$ since $\boldsymbol{\epsilon}^{C}{}_{(A}\boldsymbol{\mathcal{D}}_{B)}\boldsymbol{\mathcal{D}}_{C}\boldsymbol{Y}^{\ell m}$ vanishes for these values of ℓ.) Next, consider the gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega L^a$ where $\boldsymbol{T} :\breve{=} -\boldsymbol{X}$ and $\boldsymbol{L}_a :\breve{=} -\boldsymbol{D}_a \boldsymbol{X}$. This transformation leaves ${}^{(1)}\!\boldsymbol{\beta}_a$ and ${}^{(1)}\!\boldsymbol{\psi}$ unchanged and kills the even parity part of ${}^{(1)}\!\boldsymbol{\chi}^{TT}_{ab}$:

$$ {}^{(1)}\!\boldsymbol{\chi}^{TT}_{ab} \breve{=} \sum_{\ell \neq 0} \boldsymbol{b}^{(-)}_{\ell m}\, {}^{(1)}\!\boldsymbol{\chi}^{TT(-)\ell m}_{ab}. \tag{31} $$

To summarize, the solutions of the Einstein equation (10) is given by

$$ \begin{aligned} {}^{(1)}\!\overline{\boldsymbol{F}} &\breve{=} \sum \boldsymbol{a}_{\ell m}\, {}^{(1)}\!\boldsymbol{F}^{\ell m}, & {}^{(1)}\!\overline{\boldsymbol{\beta}}_a &\breve{=} 0, \\ {}^{(1)}\!\overline{\boldsymbol{\psi}} &\breve{=} -\frac{1}{2}\sum \boldsymbol{a}_{\ell m}\, {}^{(1)}\!\boldsymbol{F}^{\ell m} \quad \text{and} & {}^{(1)}\!\overline{\boldsymbol{\chi}}_{ab} &\breve{=} \sum_{\ell \neq 0} \boldsymbol{b}^{(-)}_{\ell m}\, {}^{(1)}\!\boldsymbol{\chi}^{TT(-)\ell m}_{ab} \end{aligned} \tag{32} $$

in the suitably chosen Poisson gauge. This is the structure of the gravitaional field common to all the ATI-1 spacetimes, and thus we define *the first order asymptotic structure* as follows.

DEFINITION:
${}^{(1)}\!\hat{g}_{ab}$ given by eq.(9) is called the *first order asymptotic structure* of an AFTI-1 spacetime, where ${}^{(1)}\!\boldsymbol{F}$, ${}^{(1)}\!\boldsymbol{\beta}_a$, ${}^{(1)}\!\boldsymbol{\psi}$ and ${}^{(1)}\!\boldsymbol{\chi}_{ab}$ takes the form eq.(32) on $\breve{\imath}^+$, in the Poisson gauge.

In such a spacetime, it can be calculated that

$$ {}^{(1)}\!\boldsymbol{E}_{ab} \breve{=} \frac{1}{2}(\boldsymbol{D}_a\boldsymbol{D}_b - \boldsymbol{h}_{ab})\, {}^{(1)}\!\boldsymbol{F} \qquad \text{and} \qquad {}^{(1)}\!\boldsymbol{B}_{ab} \breve{=} \frac{1}{2}\,\boldsymbol{\epsilon}_{ra}{}^{s}\boldsymbol{D}^{r}\, {}^{(1)}\!\boldsymbol{\chi}_{bs} \tag{33} $$

and that ${}^{(n)}\!\boldsymbol{E}_{ab}$ and ${}^{(n)}\!\boldsymbol{B}_{ab}$ for $n \geq 2$ depend on the behavior of the higher order energy-momentum tensor, which is arbitrary in an ATI-1 spacetime. (See [3] for the details of the calculation.) Eq.(33) clearly shows that ${}^{(1)}\!\boldsymbol{F}$ of the first order asymptotic structure forms the first order term of the electric part of the Weyl tensor and that ${}^{(1)}\!\boldsymbol{\chi}_{ab}$ forms the magnetic part. In other words, if we specify the sets of coefficients $\{\boldsymbol{a}_{\ell m}\}$ and $\{\boldsymbol{b}_{\ell m}\}$, the first order terms of the electric part and the magnetic part are determined, respectively.

IV. ASYMPTOTIC STATIONARITY

In this section, we introduce the notion of *asymptotic stationarity* of ATI-n spacetimes, and prove that an ATI-1 spacetime that is asymptotically stationary to order 2 must be an asymptotically Schwarzschild spacetime. The plan of the proof is: 1) we derive, in the Poisson gauge, the reduced first asymptotic structure of an ATI-1 spacetime that is asymptotically stationary to order 2 in the lemma; 2) we perform a suitable gauge transformation as to show explicitly that such an asymptotic structure approaches asymptotically the Schwarzschild metric in the theorem.

A Killing vector field $\hat{\xi}^a$ is a vector field with respect to which the Lie derivative of the metric vanishes, $\pounds_{\hat{\xi}}\hat{g}_{ab} = 0$. This fact motivates us to define an asymptotic Killing vector field and its order as follows.

DEFINITION: An ATI-m spacetime $(\hat{\mathcal{M}}, \hat{g}_{ab})$ is said to admit an *asymptotic Killing field* $\hat{\xi}^a$ to order n if

$$ \lim_{\to \breve{\imath}^+} \Omega^{-n}(\pounds_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}} = 0 \tag{34} $$

where $(\pounds_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}} := \Psi^*((\hat{e}_\mu)^a(\hat{e}_\nu)^b\pounds_{\hat{\xi}}\hat{g}_{ab})$.

Next, we consider how to define asymptotic stationarity, using this notion of an asymptotic Killing vector field. A vector field $\hat{\xi}^a$ is said to be a stationary Killing vector field in an asymptotically flat spacetime, if $\hat{\xi}^a$ is a timelike Killing field and satisfies $\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b = -1$ at infinity. Hence, we define its asymptotic correspondace as follows.

DEFINITION: A vector field ξ^a is said to be an asymptotic stationary Killing vector field to order n of an ATI-m spacetime if $\hat{\xi}^a$ is admitted as an asymptotic Killing vector field to order n in the ATI-m spacetime and satisfies

$$\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b \breve{=} -1 \tag{35}$$

on $\breve{\imath}^+$.

It is important to note that the definition of the asymptotic stationary Killing vector field implies that the leading term of $\hat{\xi}^{\hat{\mu}} = \hat{\xi}^a(\hat{e}_\mu)_a$ is of the order Ω^0. Hence, $\xi^{\hat{\mu}} := \hat{\xi}^{\hat{\mu}}$ admits a smooth limit to $\breve{\imath}^+$ and can be expanded as

$$\xi^{\hat{\mu}} = \sum_{n=1}^{\infty} {}^{(n)}\xi^{\hat{\mu}}\,\Omega^n. \tag{36}$$

(Here, ${}^{(n)}\xi^{\hat{0}}$ and ${}^{(n)}\xi^{\hat{\imath}}$ correspond to ${}^{(n+2)}\xi^{\hat{0}}$ and ${}^{(n+1)}\xi^{\hat{\imath}}$ in [3]. We do not use the notation of [3] for $\hat{\xi}^a$, because the above notation reflects the nature of the completion more intrinsically.) Simple calculation shows that the function $(\pounds_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ admits a smooth limit to $\breve{\imath}^+$, and thus can be expanded in the manner described by eq.(1).

Now we are ready to prove the following lemma.

LEMMA: An ATI-1 spacetime is asymptotically stationary to order 2 if and only if $\boldsymbol{a}_{\ell m}\breve{=}0$ for $\ell \neq 0$ and $\boldsymbol{b}_{\ell m}\breve{=}0$.

Proof of only if: If an ATI-1 spacetime is asymptotically stationary to order 2,

$$^{(n)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}\breve{=}0 \quad \text{for } n \leq 2, \tag{37}$$

$$\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b \breve{=} -1 \tag{38}$$

hold. First, we note that eq.(38) is equivalent to

$$-\,({}^{(0)}\boldsymbol{\xi}^{\hat{0}})^2 + {}^{(0)}\boldsymbol{\xi}^{a}\,{}^{(0)}\boldsymbol{\xi}_a \breve{=} -1 \tag{39}$$

where ${}^{(\ell)}\boldsymbol{\xi}^a := {}^{(\ell)}\boldsymbol{\xi}^{\hat{K}}(\boldsymbol{e}_{K})^a$ and $\{(\boldsymbol{e}_{I})_a\}_{I=1,2,3}$ is a triad of $\boldsymbol{h}_{ab}$. Second, we simplify the $n \leq 1$ part of eq.(37). In an ATI-1 spacetime, the $O(\Omega^0)$ and $O(\Omega^1)$ terms of $(\pounds_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ are given by

$$\begin{aligned}
&{}^{(0)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{0}\hat{0}}\breve{=}0, \qquad {}^{(0)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{0}\hat{\imath}}\breve{=}0, \qquad {}^{(0)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{\imath}\hat{\jmath}}\breve{=}0,\\
&{}^{(1)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{0}\hat{0}}\breve{=}0, \quad {}^{(1)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{0}\hat{\imath}}\breve{=}\left[{}^{(0)}\boldsymbol{\xi}_a - \boldsymbol{D}_a{}^{(0)}\boldsymbol{\xi}^{\hat{0}}\right](\boldsymbol{e}_{I})^a, \quad \text{and} \quad {}^{(1)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{\imath}\hat{\jmath}}\breve{=}\left[\boldsymbol{D}_{(a}{}^{(0)}\boldsymbol{\xi}_{b)} - {}^{(0)}\boldsymbol{\xi}^{\hat{0}}\boldsymbol{h}_{ab}\right](\boldsymbol{e}_{I})^a(\boldsymbol{e}_{J})^b.
\end{aligned}$$

Hence, the $n \leq 1$ part of eq.(37) is equivalent to

$$^{(0)}\boldsymbol{\xi}_a - \boldsymbol{D}_a{}^{(0)}\boldsymbol{\xi}^{\hat{0}}\breve{=}0 \quad \text{and} \quad \boldsymbol{D}_{(a}{}^{(0)}\boldsymbol{\xi}_{b)} - {}^{(0)}\boldsymbol{\xi}^{\hat{0}}\boldsymbol{h}_{ab}\breve{=}0. \tag{40}$$

Solving eqs.(39)(40) simultaneously, we find that

$$^{(0)}\boldsymbol{\xi}^{\hat{0}}\breve{=}\cosh\boldsymbol{\chi} \quad \text{and} \quad {}^{(0)}\boldsymbol{\xi}^{\hat{K}}(\boldsymbol{e}_{K})^a\breve{=}\sinh\boldsymbol{\chi}(\partial_\chi)^a. \tag{41}$$

Next, we consider the $n = 2$ part of eq.(37). Let us choose the Poisson gauge in which the first order asymptotic structure takes the form eq.(32) in an ATI-1 spacetime. Because

$$\begin{aligned}
&{}^{(2)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{0}\hat{0}}\breve{=}-{}^{(1)}\boldsymbol{\xi}^{\hat{0}} + \frac{1}{2}\left({}^{(0)}\boldsymbol{\xi}^{\hat{0}} + {}^{(0)}\boldsymbol{\xi}^m\boldsymbol{D}_m\right){}^{(1)}\boldsymbol{F} \qquad \text{and}\\
&{}^{(2)}(\pounds_{\hat{\xi}}\hat{g})_{\hat{0}\hat{\imath}}\breve{=}\left[2{}^{(1)}\boldsymbol{\xi}_a - \boldsymbol{D}_a{}^{(1)}\boldsymbol{\xi}^{\hat{0}} + \left(\boldsymbol{D}_a{}^{(0)}\boldsymbol{\xi}^{\hat{0}} - {}^{(0)}\boldsymbol{\xi}_a\right){}^{(1)}\boldsymbol{F} - 2{}^{(0)}\boldsymbol{\xi}_m{}^{(1)}\boldsymbol{\chi}_a{}^m\right](\boldsymbol{e}_{I})^a,
\end{aligned} \tag{42}$$

the $n = 2$ part of eq.(37) for $\hat{\mu} = \hat{0}$ is equivalent to

$$^{(1)}\boldsymbol{\xi}^{\hat{0}}\breve{=}\frac{1}{2}\left({}^{(0)}\boldsymbol{\xi}^{\hat{0}} + {}^{(0)}\boldsymbol{\xi}^m\boldsymbol{D}_m\right){}^{(1)}\boldsymbol{F} \quad \text{and} \quad {}^{(1)}\boldsymbol{\xi}^{\hat{K}}(e_{K})^a\breve{=}\frac{1}{2}\left[\boldsymbol{D}_a{}^{(1)}\boldsymbol{\xi}^{\hat{0}} - \left(\boldsymbol{D}_a{}^{(0)}\boldsymbol{\xi}^{\hat{0}} - {}^{(0)}\boldsymbol{\xi}_a\right){}^{(1)}\boldsymbol{F} + 2{}^{(0)}\boldsymbol{\xi}_m{}^{(1)}\boldsymbol{\chi}_a{}^m\right]. \tag{43}$$

Using eqs.(41)(43), we find that ${}^{(2)}\!(\pounds_{\hat\xi}\hat g)_{\hat\imath\hat\jmath}$ are given by

$$ {}^{(2)}\!(\pounds_{\hat\xi}\hat g)_{\hat\imath\hat\jmath} \,\breve{=}\, 2\sinh\boldsymbol{\chi}\left[{}^{1}\!\boldsymbol{\mathcal{L}}_{ab}\,{}^{(1)}\!\boldsymbol{F}+{}^{2}\!\boldsymbol{\mathcal{L}}_{(a}{}^{m}\,{}^{(1)}\!\boldsymbol{\chi}_{b)m}\right](\boldsymbol{e}_{\mathrm{I}})^{a}(\boldsymbol{e}_{\mathrm{J}})^{b} \tag{44}$$

where the derivative operators ${}^{1}\!\boldsymbol{\mathcal{L}}_{ab}$ and ${}^{2}\!\boldsymbol{\mathcal{L}}_{ab}$ are given by

$$ {}^{1}\!\boldsymbol{\mathcal{L}}_{ab}:\breve{=}\frac{3}{\tanh\boldsymbol{\chi}}(\boldsymbol{D}_a\boldsymbol{D}_b-\boldsymbol{h}_{ab})+(\partial_{\chi})^{m}\boldsymbol{D}_{(a}\boldsymbol{D}_{b)}\boldsymbol{D}_m-(d\boldsymbol{\chi})_{(a}\boldsymbol{D}_{b)} \quad\text{and}\quad {}^{2}\!\boldsymbol{\mathcal{L}}_{ab}:\breve{=}2(d\boldsymbol{\chi})_p\boldsymbol{h}_a{}^{[p}\boldsymbol{h}_b{}^{m]}\boldsymbol{D}_m, \tag{45}$$

respectively. By the definition of ${}^{(1)}\!\boldsymbol{F}$ and ${}^{(1)}\!\boldsymbol{\chi}_{ab}$, ${}^{(2)}\!(\pounds_{\hat\xi}\hat g)_{\hat\imath\hat\jmath}$ vanishes if and only if ${}^{1}\!\boldsymbol{\mathcal{L}}_{ab}\,{}^{(1)}\!\boldsymbol{F}$ and ${}^{2}\!\boldsymbol{\mathcal{L}}_{(a}{}^{s}\,{}^{(1)}\!\boldsymbol{\chi}_{b)s}$ vanish independently. With the help of eqs.(16)(32), ${}^{1}\!\boldsymbol{\mathcal{L}}_{ab}\,{}^{(1)}\!\boldsymbol{F}\breve{=}0$ can be rewritten as

$$\begin{aligned}\sum\boldsymbol{a}_{\ell m}\Bigg[&\left(\boldsymbol{T}_0^{\ell\,\prime\prime\prime}+\frac{3\boldsymbol{T}_0^{\ell\,\prime\prime}}{\tanh\boldsymbol{\chi}}-\boldsymbol{T}_0^{\ell\,\prime}-\frac{3\boldsymbol{T}_0^{\ell}}{\tanh\boldsymbol{\chi}}\right)(d\boldsymbol{\chi})_a(d\boldsymbol{\chi})_b+2\left(\boldsymbol{T}_0^{\ell\,\prime\prime}-\frac{\boldsymbol{T}_0^{\ell\,\prime}}{\tanh\boldsymbol{\chi}}-(5+\frac{4}{\sinh^2\boldsymbol{\chi}})\boldsymbol{T}_0^{\ell}\right)(d\boldsymbol{\chi})_{(a}\boldsymbol{\mathcal{D}}_{b)}\\&+\sinh\boldsymbol{\chi}\cosh\boldsymbol{\chi}\left(\boldsymbol{T}_0^{\ell\,\prime\prime}+\frac{2\boldsymbol{T}_0^{\ell\,\prime}}{\tanh\boldsymbol{\chi}}-3\boldsymbol{T}_0^{\ell}-\frac{\ell(\ell+1)}{2\sinh\boldsymbol{\chi}\cosh\boldsymbol{\chi}}(\boldsymbol{T}_0^{\ell\,\prime}+\frac{\boldsymbol{T}_0^{\ell}}{\tanh\boldsymbol{\chi}})\right)(d\boldsymbol{\sigma})_{ab}\\&+\left(\boldsymbol{T}_0^{\ell\,\prime}+\frac{\boldsymbol{T}_0^{\ell}}{\tanh\boldsymbol{\chi}}\right)(\boldsymbol{\mathcal{D}}_a\boldsymbol{\mathcal{D}}_b-\frac{1}{2}(d\boldsymbol{\sigma})_{ab}\boldsymbol{\mathcal{D}}^c\boldsymbol{\mathcal{D}}_c)\Bigg]\boldsymbol{Y}^{\ell m}\breve{=}0.\end{aligned} \tag{46}$$

We first consider the $\ell\neq0$ terms of the above equation. Noting that all the components are independent and using eqs.(17)(19), it is found that *if* $\boldsymbol{a}_{\ell m}\breve{\neq}0$ the above equation (46) is equivalent to

$$\frac{\ell(\ell+1)}{\sinh^2\boldsymbol{\chi}}(\boldsymbol{\mathcal{P}}_2^{\ell\,\prime}+\frac{\boldsymbol{\mathcal{P}}_2^{\ell}}{\tanh\boldsymbol{\chi}})\breve{=}0,\quad\left(\frac{\ell^2+\ell-4}{\sinh^2\boldsymbol{\chi}}-2\right)\boldsymbol{\mathcal{P}}_2^{\ell}-3\boldsymbol{\mathcal{P}}_2^{\ell\,\prime}\breve{=}0,\quad\frac{\ell(\ell+1)}{\sinh^2\boldsymbol{\chi}}\boldsymbol{\mathcal{P}}_2^{\ell}\breve{=}0\quad\text{and}\quad\boldsymbol{\mathcal{P}}_2^{\ell\,\prime}+\frac{\boldsymbol{\mathcal{P}}_2^{\ell}}{\tanh\boldsymbol{\chi}}\breve{=}0. \tag{47}$$

Apparently, there is no integer ℓ that is not equal to 0 and that satisfies eqs.(47), simultaneously. Therefore, $\boldsymbol{a}_{\ell m}\breve{=}0$ for $\ell\neq0$. Noting that $\boldsymbol{\mathcal{D}}_a\boldsymbol{Y}^{\ell m}$ vanishes for $\ell=0$, we find that all the components of the $\ell=0$ terms of the right-hand side of eq.(46) vanishes. Therefore, $\boldsymbol{a}_{00}$ can take arbitrary value. Next consider ${}^{2}\!\boldsymbol{\mathcal{L}}_{(a}{}^{s}\,{}^{(1)}\!\boldsymbol{\chi}_{b)s}\breve{=}0$. This equation is satisfied if and only if

$$\sum_{\ell\neq0}\boldsymbol{b}_{\ell m}\left[(\boldsymbol{T}_1^{\ell\,\prime}+\frac{\boldsymbol{T}_1^{\ell}}{\tanh\boldsymbol{\chi}})(d\boldsymbol{\chi})_{(a}\boldsymbol{\epsilon}_{b)}{}^{r}\boldsymbol{\mathcal{D}}_r+(\boldsymbol{T}_2^{\ell\,\prime}-\frac{\boldsymbol{T}_2^{\ell}}{\tanh\boldsymbol{\chi}}-\boldsymbol{T}_1^{\ell})\boldsymbol{\epsilon}^{r}{}_{(a}\boldsymbol{\mathcal{D}}_{b)}\boldsymbol{\mathcal{D}}_r\right]\boldsymbol{Y}^{\ell m}\breve{=}0. \tag{48}$$

With the same reasoning, we find that if $\boldsymbol{b}_{\ell m}\breve{=}0$ the above equation (48) is equivalent to

$$\frac{\ell(\ell+1)}{\sinh^2\boldsymbol{\chi}}\boldsymbol{\mathcal{P}}_0^{\ell}\breve{=}0\qquad\text{and}\qquad\frac{(\ell+2)(\ell-1)}{\sinh^2\boldsymbol{\chi}}\boldsymbol{\mathcal{P}}_0^{\ell}\breve{=}0 \tag{49}$$

and that there is no ℓ that satisfies the above equations simulataneously. Hence, we conclude $\boldsymbol{b}_{\ell m}\breve{=}0$.□

Proof of if: If $\boldsymbol{a}_{\ell m}\breve{=}0$ for $\ell\neq0$ and $\boldsymbol{b}_{\ell m}=0$ in an ATI-1 spacetime, the vector field $\hat\xi^a$ whose $O(\Omega^0)$ and $O(\Omega^1)$ terms are given by eqs.(41)(43) satisfy ${}^{(n)}\!(\pounds_{\hat\xi}\hat g)_{\hat\mu\hat\nu}\breve{=}0$ for $n\leq2$ and $\hat g_{ab}\hat\xi^a\hat\xi^b\breve{=}-1$. Hence, $\hat\xi^a$ is an asymptotically stationary Killing vector field to order 2.□

The fact that $\boldsymbol{a}_{\ell m}\breve{=}0$ for $\ell\neq0$ and $\boldsymbol{b}_{\ell m}\breve{=}0$ means that in such an ATI-1 spacetime, the first asymptotic structure takes the simple form

$$ {}^{(1)}\!\boldsymbol{F}\breve{=}\boldsymbol{a}_{00}\,{}^{(1)}\!\boldsymbol{F}^{00},\qquad{}^{(1)}\!\boldsymbol{\psi}\breve{=}-\frac{1}{2}\boldsymbol{a}_{00}\,{}^{(1)}\!\boldsymbol{F}^{00},\qquad{}^{(1)}\!\boldsymbol{\beta}\breve{=}0\qquad\text{and}\qquad{}^{(1)}\!\boldsymbol{\chi}_{ab}\breve{=}0. \tag{50}$$

Before we show that such an ATI-1 spacetime is an asymptotically Schwarzschild spacetime, we remark an important fact relating to the definition of the angular-momentum of an asymptotically flat spacetime. To define angular-momentum, one must impose the condition that the $O(\Omega^1)$ term of the magnetic part of the Weyl tensor vanish [2–4]. However, the physical meaning of the condition was left unclear. The lemma and eqs.(33)(50) tells us that the meaning is that the angular-momentum of an asymptotically spacetime can be defined if the spacetime is asymptotically stationary to order 2.

Theorem: An ATI-1 spacetime which is asymptotically stationary to order 2 is an asymptotically Schwarzschild spacetime with mass a_{00} in the sense that the metric takes the form

$$\hat{g}_{ab} = \hat{g}^{\mathsf{SCH}}_{ab} + O(r^{-2}) + O(t^{-2}) \qquad \text{where} \qquad \hat{g}^{\mathsf{SCH}}_{ab} := -(1 - \frac{2a_{00}}{r})(dt)_a(dt)_b + (1 - \frac{2a_{00}}{r})^{-1}(dr)_a(dr)_b + r^2(d\sigma)_{ab} \tag{51}$$

in which $t > r$.

Proof: From the lemma, in an ATI-1 spacetime which is asymptotically stationary to order 2, $\boldsymbol{a}_{\ell m} = 0$ for $\ell \neq 0$ and $\boldsymbol{b}_{\ell m} = 0$ hold. Thus, the first order term of the metric ${}^{(1)}\hat{g}_{ab}$ takes the form

$$ {}^{(1)}\hat{g}_{ab} = (e^{-\eta})^2 \left[a_{00} {}^{(1)}F^{00}(d\eta)_a(d\eta)_a + a_{00} {}^{(1)}F^{00}h_{ab} \right]. \tag{52}$$

Under a gauge transformation generated by $\xi^a = \Omega T(\partial_\eta)^a + \Omega D^a L$ where

$$\boldsymbol{T} :\stackrel{\vee}{=} \boldsymbol{a}_{00}(2\boldsymbol{\chi}\cosh\boldsymbol{\chi} + \sinh\boldsymbol{\chi}) \qquad \text{and} \qquad \boldsymbol{L} :\stackrel{\vee}{=} -4\boldsymbol{a}_{00}\sinh\boldsymbol{\chi} + \boldsymbol{T}, \tag{53}$$

${}^{(1)}\hat{g}_{ab}$ transforms as

$$ {}^{(1)}\hat{g}_{ab} \mapsto (e^{-\eta})^2 \left[a_{00} {}^{(1)}F^{00}(d\eta)_a(d\eta)_a - 8a_{00}\cosh\chi(d\eta)_{(a}(d\chi)_{b)} + a_{00} {}^{(1)}F^{00}(d\chi)_a(d\chi)_b \right]. \tag{54}$$

Then, the change of variables, $t = \Omega^{-1}\cosh\chi$ and $r = \Omega^{-1}\sinh\chi$, leads us to

$$O(\Omega^2) = \sinh^{-2}\chi O(\Omega^2) + \cosh^{-2}\chi O(\Omega^2) = O(r^{-2}) + O(t^{-2}), \qquad \frac{r}{t} = \tanh\chi < 1. \tag{55}$$

and

$$ {}^{(0)}\hat{g}_{ab} + \Omega {}^{(1)}\hat{g}_{ab} = \hat{g}^{\mathsf{SCH}}_{ab} + O(\Omega^2). \tag{56}$$

Hence, eq.(51) holds.□

It is important to note here that asymptotically Schwarzschild spacetimes, that are defined in eq.(51), comprise the Kerr spacetime also. This can be understood by writing the Kerr metric $\hat{g}^{\mathsf{KER}}_{ab}$ with the coordinates (Ω, χ) where $t = \cosh\chi$ and $r = \sinh\chi$:

$$\hat{g}^{\mathsf{KER}}_{ab} = \hat{g}^{\mathsf{SCH}}_{ab} + \Omega^2\, {}^{(2)}\hat{g}_{ab} + O(\Omega^3) \tag{57}$$

where

$$\begin{aligned} {}^{(2)}\hat{g}_{ab} = {}& (4m^2 - a^2\sin^2\theta)\left((d\eta)_{(a}(d\eta)_{b)} - 2\frac{(d\chi)_{(a}(d\eta)_{b)}}{\tanh\chi}\right) + 4ma\frac{\sin^2\theta}{\tanh\chi}(d\phi)_{(a}(d\eta)_{b)} \\ &+ \frac{4m^2 - a^2\sin^2\theta}{\tanh\chi}(d\chi)_a(d\chi)_b - 4am\sin^2\theta^2(d\phi)_{(a}(d\chi)_{b)} + a^2\cos^2\theta(d\theta)_a(d\theta)_b + a^2\sin^2\theta(d\phi)_a(d\phi)_b. \end{aligned} \tag{58}$$

In other words, the Kerr spacetime is a special spacetime of asymptotically Schwarzschild spacetimes, which possesses a particular second-order asymptotic structure, i.e., ${}^{(2)}\hat{g}_{ab}$ given by eq.(58).

V. SUMMARY AND REMARKS

In this paper, we have proved that an asymptotically flat spacetime as defined in definition 1 of sec.II is an asymptotically Schwarzschild spacetime in the sense of eq.(51), if the energy-momentum of the spacetime falls off at the rate faster than $O(\Omega^3)$ and the spacetime is asymptotically stationary to order 2 in the sense that $(\pounds_{\hat{\xi}}\hat{g})_{\hat{\mu}\hat{\nu}}$ falls off at the rate faster than $O(\Omega^2)$ for the asymptotically timelike vector $\hat{\xi}^a$, $\hat{g}_{ab}\hat{\xi}^a\hat{\xi}^b \stackrel{\vee}{=} -1$.

Finally, we give a remark. Although we have solved the Einstein equation (10) and obtained the first order asymptotic structure, we did not impose any physically-suitable boundary conditions on the solutions. Hence, the obtained first order asymptotic structure may include those that are *unphysical*. For example, a solution that describes an incoming gravitational wave from future null infinity or an outcoming wave from the event horizon. In other words, *physically* acceptable gravitational fields around a black hole may be obtained only after a suitable boundary condition

are imposed on the solutions. The derivation of the physical first order asymptotic structure may be profitable because it may be possible to show that an ATI-1 spacetime with such a physical structure is intrinsically asymptotically stationary to order 1, and thus is an asymptotically Schwarzschild spacetime. This result is anticipated because we expet that a spacetime with a black hole that becomes vacuum also becomes stationary, due to the nature of the black hole. This is an important point that should be clarified.

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